Self-Organizing Innovation Networks: When do Small Worlds Emerge?

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Abstract

In this paper, we present a model of 'collective innovation' built upon the network formation formalism. In our model, agents localized on a circle benefit from knowledge flows from other agents with whom they are directly or indirectly connected. They support costs for direct connections which are linearly increasing with geographic distance. The dynamic process of network formation exhibits preferential meeting for close agents (in the relational network and in the geographic metrics). We show how the set of stochastically stable networks selected in the long run is affected by the degree of knowledge transferability. We find critical values of this parameter for which stable "small world" networks are dynamically selected.

Classification Codes: C62, C63, C70, L20, O31, R10

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1. Introduction

A growing body of empirical literature is concerned with the influence of network relations on firms rates of innovation. Far from being the outcome of isolated agents efforts, innovation is usually described as a collective and interactive process (e.g. von Hippel, 1989). Allen (1983) puts in evidence those characteristics focusing on what he calls the 'collective invention' phenomenon. According to him, it occurs when social interactions generate knowledge disclosure between agents belonging to competing firms which in turn stimulates incremental innovation. Powell *et al.* (1996) also emphasize the role of networks as a source of innovation, improving and facilitating information and knowledge transfers: "a network serves as a locus of innovation because it provides timely access to knowledge and resources that are otherwise unavailable" (Powell *et al.*, 1996).

Another body of empirical analyses highlights that the "local milieu" plays an important role in favoring knowledge diffusion (Antonelli, 1999). Many empirical studies show that innovation activities are spatially clustered and benefit from localized knowledge spillovers (Feldman, 1994; Audretsch, 1998). Saxenian (1994) shows that in the Silicon Valley, it is a dense social network stimulated by an open local labor market which promotes collective learning among competing firms through collaborations or informal communication. Nevertheless, making use of a network approach, she provides qualitative evidence showing that, despite similar origins and technologies, Silicon Valley and Route 128 followed very distinct evolution trajectories. The example of Route 128 demonstrates that geographic clustering is not a sufficient condition to ensure the emergence of regional networks. Thus, geographic proximity can only be considered as an imperfect proxy to capture the existence of network relations which generate knowledge spillovers (Breschi and Lissoni, 2003). On the other way on, geographic space should be considered as non neutral for the process of network formation: thus the "milieu" and the network (when it exists) are likely to overlap.

This paper aims to study 'collective innovation' in a model of network formation, where myopic self-interested agents benefit from knowledge flows from agents with whom they interact either directly or indirectly. We make the assumption that the higher the distance in the relational network the weaker the spillover. In other words, we consider that a decay is affecting knowledge diffusion. The agents support costs for direct connections which are linearly increasing with geographic distance separating them. The main concern is the dynamic formation of networks. Indeed, if several previous theoretical works focus on how network structures matter for innovation dynamics through information, knowledge or technology diffusion (for example, David and Foray, 1994; Valente, 1996; Cowan and Jonard, 2001; Young, 2002), they are not concerned with network formation which remains a crucial issue for knowledge dynamics and innovation[1]. This question is of interest: if the network structure has obviously much to say about innovative performance, then one may naturally wonder about the circumstances that allow various network structures to emerge. Thus, a "self-organization" perspective[2] is chosen to study the emergence of networks and its co-evolution with knowledge diffusion. In particular, we focus on how network selection is affected by the easiness of knowledge flows (i.e. by the decay).

Directly related to this work are the recent formal economic contributions highlighting how (both individual and collective) behaviors and performances are grounded in networks, which are often in turn shaped by agents[3]. The very originality of this approach is the focus on network formation. A theoretical framework has been proposed by Jackson and Wolinski (1996) based on a two-sided network formation game[4]. Concepts of (myopic and pairwise) stability and of efficiency have been introduced in this framework. This contribution also constitutes an important point of departure to analyze and model endogenously emerging structures.

Jackson and Watts (2002) (initiated in Watts, 2001) have developped the dynamics of that approach by introducing the notion of stochastically stable networks based on notions and results initially proposed by Young (1993) and Kandori *et al.* (1993). To study the dynamic formation of networks, we make use of their stochastic Markov process. More precisely, they introduced random errors which invert agents' right decisions in creating, maintaining or deleting links. While following their contribution, our model departs from theirs in that we enrich the meeting process. We introduce a *preferential meeting* process which governs the dynamic process of links formation, assuming that agents meet easily other agents in their neighborhood. This way we simply reject the uniform meeting probability and weight the probability that two unconnected agents meet with both the inverse of their distance (on both metrics: the relational and the geographic ones). We expect that the underlying Markov chain will select pairwise equilibria that have some in common with the empirical literature on networks.

[1] Cowan and Jonard (2001) study the impact of network architectures on knowledge diffusion and show that knowledge grows at different rates depending on them. This model has been extended in Cowan *et al.* (2002) who model agents matching with each other to combine their knowledge to innovate. Nevertheless, the networks formation is only captured by cumulative frequency matrices which constitute the "trace" of ponctual interactions.

[2] See for example Lesourne and Orléan (1998) and Paulré (1997) for some insights on the concept of "self-organization".

[3] Predictions concern various contexts such as information diffusion on job opportunities (Calvó-Armengol and Jackson, 2001; Tesfatsion, 2001; Calvó-Armengol, 2003), firms' design (Radner, 1993; Bolton and Dewatripont, 1996, Guimerà et al., 2001), R&D collaborations (Goyal and Joshi, 2000; Goyal and Moraga, 2001), market organization (Weisbuch *et al.*, 2000), etc.

[4] Their approach is also usually called "mixed approach" since it is half way between the cooperative (Slikker and van den Nouweland, 2000) and the non-cooperative ones (Bala and Goyal, 2000). More precisely, it assumes that two agents have to agree simultaneously to become directly connected while only one defection breaks an existing link.

This way we come to another body of literature which emerged recently in Physics dealing with the structure of large networks as evidenced by web sites links, relational networks, coauthoring scientific papers (Barabási and Albert, 1999, 2000; Watts and Strogatz 1998; Newman *et al.*, 2001)[5]. The general conclusions of this literature are that such networks are highly clustered and exhibit some long distant connections[6][7]. Such structures are usually called *small worlds* because despite a very large number of nodes (agents) the average distance between them is usually small (known as the "six degree of separation", Milgram, 1963).

Doing this, we finally face another issue, namely the characterization of equilibrium networks for which the economic literature on network formation has not dedicated much attention, focusing mainly on the compatibility between networks efficiency and stability (Jackson, 2003). The typical network structures discussed in this framework are the cycle, the empty network, the star, and the complete network. Here, the selected equilibria we obtain cannot fall anymore systematically under these usual categories. Therefore, we make use of several indexes that capture interesting features of the graphs. We show that different values of the knowledge transferability parameter generate qualitatively different network architectures. In particular we find that for critical values of the decay in knowledge spillover parameter the stochastic process selects pairwise stable small worlds networks in the long run.

The paper is organized as follows. Section 2 presents the static features of our 'collective innovation' model in the network formation formalism and introduces indexes used to characterize networks. Section 3 is devoted to the dynamics, highlighting the preferential meeting rule we propose and the generic properties of the stochastic process. The results obtained are presented in Section 4. The last section concludes.

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[5] As a matter of fact, our preferential meeting rule has some in common with the so called "preferential attachment" process which has recently been highlighted as crucial for generating networks characterized by skew vertices distribution. Several models have been introduced: Barabási *et al.* (2001), Yook *et al.* (2001), Jeong *et al.* (2003), Bianconi and Barabási (2001), Newman (2001) and, for a complete review, Albert and Barabási (2002). However these models propose a poor description of agents behaviors.

[6] Those non local, distant connections can be viewed as "weak ties" as described by Granovetter (1973).

[7] These two features characterize small worlds *à la* Watts and Strogatz (1998). Another general result of that literature is that the nodes degree distribution is usually quite skew. This last result comes from the *scale free networks* approach (Barabási and Albert, 1999, 2000). In this paper, we do not concentrate on that issue which should be extensively studied in a forthcoming dedicated work.

2. The 'collective innovation' model in the network formation formalism

The point of departure of our model is the network formation formalism introduced by Jackson and Wolinski (1996). Consider a finite set of n agents, $N = \{1, 2, ..., n\}$ with $n \geq 3$, and let i and j be two members of this set. Each agent is assumed to increase its knowledge through internal capacities and/or by communicating directly through costly relationships with other agents. Direct connections between agents, which are called *pairwise links* since the willingness of both the two agents is necessary to establish and maintain a link, form the relational network which is represented as a non-directed graph. In this model, agents can also benefit from indirect (and costless) connections, through the relational network of their partners, but in a decreasing manner *i.e.* the benefits deteriorate with the *relational distance*. We then consider that the rate at which agents innovate is deduced from their knowledge accumulation rate which is in turn obtained through their relations network.

We begin with some basic notions in network formation. We then turn to the description of the innovation process through knowledge diffusion and accumulation resulting from internal capacities and from direct and indirect connections that allow the absorption of knowledge. Finally, we introduce several graph indexes which can be used to characterize various networks architectures.

2.1. Basic notions in network formation

2.1.1. Properties and typical structures of graphs

Consider a finite set of n agents, $N = \{1, 2, ..., n\}$ with $n \geq 3$, and let i and j be two members of this set. Agents are represented by the nodes of a non-directed graph which edges represent the links between them. The graph constitutes the relational network between the agents. A link between two distinct agents i and $j \in N$ is denoted ij. A graph g is a list of non ordered pairs of connected and distinct agents. Formally, $\{ij\} \in g$ means that ij exists in g. We define the complete graph $g^N = \{ij \mid i, j \in N\}$ as the set of all subsets of N of size 2, where all players are connected with all the others. Let $g \subseteq g^N$ be an arbitrary collection of links on N. We define $G = \left\{g \subseteq g^N\right\}$ as the finite set of all possible graphs between the n agents.

Let $g' = g + ij = g \cup \{ij\}$ and $g'' = g - ij = g \backslash \{ij\}$ be respectively the graph obtained by adding ij and the one obtained by deleting ij to the existing graph g. The graphs g and g' are said to be *adjacent* as well as the graphs g and g''. For any g, we define $N(g) = \{i \mid \exists j : ij \in g\}$, the set of agents who have at least one link in the network g. We also define $N_i(g)$ as the set of neighbors agents i has, that is: $N_i(g) = \{j \mid ij \in g\}$. The cardinal of that set

$\eta_i(g) = \#N_i(g)$ is called the *degree* of node i. The total number of links in the graph g is $\eta(g) = \#g = \frac{1}{2}\sum_{i\in N}\eta_i(g)$, while the average number of neighbors is given by $\overline{\eta}(g) = 2\eta(g)/n$.

A *path* in a non empty graph $g \in G$ connecting i to j, is a sequence of edges between distinct agents such that $\{i_1 i_2, i_2 i_3, ..., i_{k-1} i_k\} \subset g$ where $i_1 = i$, $i_k = j$. The length of a path is the number of edges it contains. Let $i \longleftrightarrow_g j$ be the set of paths connecting i and j on graph g. The set of *shortest paths* between i and j on g noted $i \widetilde{\longleftrightarrow}_g j$ is such that if $\forall k \in i \widetilde{\longleftrightarrow}_g j$; implies that $k \in i \longleftrightarrow j$ and $\#k = \min_{h \in i \longleftrightarrow_g j} \#h$. We define the *geodesic distance* between two agents i and j as the number of links of the shortest path between them: $d(i,j) = d_g(ij) = \#k \in i \widetilde{\longleftrightarrow}_g j$. When there is no path between i and j then their geodesic distance is conventionally infinite: $d(i,j) = \infty$.

An external metric is also introduced, representing for example, the geographic position of agents (Johnson and Gilles, 2000). Such external metrics defines a new distance operator denoted $d'(i,j)$. In our model, we consider that agents are located on a circle (or a ring). Without loss of generality, agents are assumed to be ordered according to their index, such that $i \in N(i-1)$ and $i \in N(i+1)$ but agent 1 and agent n who are neighbors. The geographic distance may simply be obtained by $d'(i,j) = \min\{|i-j|\,; n - |i-j|\}$.

Finally, a graph $g \subseteq g^N$ is said to be *connected* if there exists a path between any two vertices of g. The subgraph $g' \subset g$ is a *connected component* of g, if:

- for all $i \in N(g')$ and $j \in N(g')$ with $i \neq j$, there exists a path in g' connecting i and j and,

- if $i \in N(g')$ and $j \notin N(g')$, with $i \neq j$, there doesn't exist a path in g' connecting i and j.

The set of all components of g is denoted by $C(g)$ such that: $g = \cup_{g' \in C(g)} g'$ (a component cannot consist in an isolated agent who has no links).

Several typical graphs can be described. Let $i, j \in N$. First of all, the *empty graph*, denoted $g^{\emptyset}$, is such that it does not contain any links. We call a network $g \in G$ a *ring* if g is connected and if :

- for all $i < j : ij \in g$, there does not exist h such that $i < h < j$ and

- for all $i > j : ij \in g$, there does not exist h such that $j < h < i$.

Such a graph is denoted g°. It is a regular network of order $k = 1$, in which all agents are connected and only connected with their two closest geographic neighbors. The double ring denoted $g^{2\circ}$ is a regular network of order $k = 2$ such that all agents are connected and only connected with their four closest neighbors. Finally, a non empty graph $g \in G$ is a (complete) *star*, denoted $g^{\star}$, if there exists $i \in N$ such that if $jk \in g^{\star}$, then either $j = i$ or $k = i$. Agent i is called the center of the star. Notice that there are n possible stars, since every node can be the center.

2.1.2. Network formation, stability and efficiency

Over time, pairs of agents meet and decide to form, maintain or break links. The formation of a link requires the consent of both the two agents but not its deletion which can emanate from one of them unilaterally. Moreover, agents are myopic which means that they take decisions on the basis of their impacts only on their current payoffs i.e. according to the state of the current network. Let $\pi_i(g_t)$ be the individual payoffs that agent i receives from the graph g_t. Jackson and Wolinski (1996) introduce the notion of *pairwise stability* which can be distinguished from the one of Nash equilibrium since the process of network formation is both cooperative and non cooperative. The formal definition of this notion is the following.

Definition 1. *(Jackson and Wolinski, 1996) A network $g \subseteq g^N$ is pairwise stable if:*
(i) for all $ij \in g$, $\pi_i(g) \geq \pi_i(g-ij)$ and $\pi_j(g) \geq \pi_j(g-ij)$, and
(ii) for all $ij \notin g$, if $\pi_i(g+ij) > \pi_i(g)$ then $\pi_j(g+ij) < \pi_j(g)$.

The efficiency of a network is computed by the *total value* of the corresponding graph g, which is a function $\pi : \{g \mid g \subseteq g^N\} \to \mathbb{R}$, with $\pi(\emptyset) = 0$. At a given period t, it is given by:

$$\pi(g_t) = \sum_{i \in N} \pi_i(g_t) \quad (1)$$

Definition 2. *(Jackson and Wolinski, 1996) A network $g \subseteq g^N$ is efficient if it maximizes the value function $\pi(g)$ on the set of all possible graphs $\{g \mid g \subseteq g^N\}$ i.e. $\pi(g) \geq \pi(g')$ for all $g' \subseteq g^N$.*

2.2. Knowledge flows and innovation

2.2.1. Knowledge accumulation and (memoryless) innovation

Let us assume that the arrival of innovation follows a Poisson process which is redrawn afresh at each period of the discrete time[8]. The arrival of innovations is thus exponentially distributed and its rate is assumed to be a linear function of knowledge accumulation. Such modeling is quite similar to the patent race one presented in Dasgupta and Stiglitz (1980) or in Reinganum (1989) apart from the fact that we are not interested in the 'race' dimension: The innovations are not substitute but complement. Let k_i^t denote agent i's stock of knowledge at period t. The instantaneous probability that i innovates when reaching any given level

[8] This assumption allows the Poisson process to be homogeneous and thus to preserve its "memoryless" property that is simply having a constant hazard rate. Moreover, since the exponential cumulative density function is always concave, the instantaneous innovation probability (non conditioned on the information about previous non-innovation) exhibits decreasing returns with the stock of knowledge. Thus, initializing the arrival rate at each period mitigates the consequences of that feature. A more complex and detailed version of the innovation process is proposed in Carayol and Roux (2003a).

$k \in \left[k_i^t; k_i^{t+1}\right[$ of his stock of knowledge is given by the exponential density function: $P\left(k_i = k\right) = \lambda e^{-\lambda k}$. The probability that agent i innovates reaching the knowledge level k knowing that he has not innovated already (since the last innovation on the unitary discrete period) is given by the following conditional probability: $q_i = P\left\{k_i \in [k, k+dk] \mid k_i > k\right\}$, which may be computed using the well known memoryless property of the exponential distribution (having a constant hazard rate), as follows:

$$q_i = \frac{P\left\{k_i \in [k, k+dk], k_i > k\right\}}{P\left(k_i > k\right)} = \frac{f(k)}{(1 - F(k))} = \lambda$$

Thus, the expected number of innovations generated by i during any unitary period t is given by:

$$\theta_i^t = \int_t^{t+1} q_i k_i^\tau d\tau \tag{2}$$

Having defined $\Delta k_i^t = \int_t^{t+1} k_i^\tau d\tau$ the knowledge variation over period t, one clearly gets:

$$\theta_i^t = \lambda \Delta k_i^t \tag{3}$$

2.2.2. Network and knowledge diffusion

Let us now turn toward describing how the knowledge is diffused through the network connections. Let us assume that knowledge is accumulated both through internal (fixed) capacities of the agent and through the direct and indirect connections that allow him to absorb others' (new) knowledge. Thus the total knowledge accumulated at period t may be obtained as follows:

$$\Delta k_i^t = \Delta k_i\left(g_t\right) = \omega_i + \sum_{j \in N \setminus i} \delta^{d(i,j)} \omega_j \tag{4}$$

where g_t is the state of the current network (which is invariant on $[t, t+1[$), ω_i and ω_j are respectively the knowledge created by agents $i, j \in N$ during one unitary period of time and which are assumed to be exogenous and constant over time and agents. Thus the second component of the expression (4) is traducing the flow of knowledge absorbed by i, which emanates simultaneously from other agents j (assuming no time lag for simplicity), through direct and indirect interconnections between i and agents j. Thus, parameter δ represents the transferability factor that is the share of new knowledge produced which is effectively directly or indirectly transmitted through each edge. Hence, we assume that $\delta \in \left]0, 1\right[$. For instance, if i and j are indirectly connected through a third agent, each will get δ^2 of the flow of knowledge each creates.

Let us now define the (expected) payoff function which is deduced from the shape of the graph:

$$\pi_i^t = \theta_i^t V - c_i^t \tag{5}$$

where θ_i^t is the expected number of innovations seen above (3), V is the net profit generated by an innovation and c_i^t is the costs incurred by i, computed as follows:

$$c_i^t = c_i(g_t) = C + \sum_{j:ij \in g_t} cd'(i,j) \tag{6}$$

It is thus potentially affected by a fixed cost and the costs spent for being connected to his direct neighbors[9].

The net profit generated by any agent i at period t, may be thus understood as a function of the graph and the position i occupies in it. That value may thus be written as $\pi_i^t = \pi_i(g_t) = \pi(\Delta k_i(g_t), c_i(g_t))$. Compiling expressions (5) (4) and (6) one gets:

$$\pi_i(g_t) = \lambda V \left(\omega_i + \sum_{j \in N \setminus i} \delta^{d(i,j)} \omega_j \right) - C - \sum_{j:ij \in g_t} cd'(i,j) \tag{7}$$

Remark 1. *Our formulation of the payoff function is voluntarily very close to the so-called "connections model" first introduced in Jackson and Wolinski (1996). If we arbitrarily fix $\lambda = 1/V$, $C = 0$, and $d'(i,j) = 1, \forall i, j$, thus we have the same formulation as theirs. Notice that if we have $\omega_i = \omega_j = \lambda V = C = 1$, then one gets the simple connections model which is well known in the network formation literature. One can also observe that when reintroducing geographic distance in link cost, then one obtains the same payoffs specification as the one of Johnson and Gilles (2000), who first introduced some external metric (theirs is the line instead as the circle in our model).*

2.3. Networks characterization: some indexes

Finally, we introduce several indexes which may all together contribute to improve the standard characterization of networks.

2.3.1. Expected efficiency

The social surplus generated by a network is generally computed by simply adding individual payoffs. Thus the *average social surplus* is given by:

$$\overline{\pi}(g) = \frac{1}{\#N} \sum_{i \in N} \pi_i(g) \tag{8}$$

[9] Relying on Debreu (1969) hypothesis according to which closely located players incur less cost to establish communication, Johnson and Gilles (2000) have first extended the connections model of Jackson and Wolinsky (1996) introducing a spatial cost topology in their network formation approach. Links costs are increasing with geographic distance between agents. The traditional assumption is that it's less costly to establish and maintain relationships when agents are geographicly close.

We may be also interested in the overall allocation of payoffs, thus following Cowan and Jonard (2001) we may compute the *variance in individual payoffs* as follows:

$$var(\pi) = \frac{1}{\#N} \sum_{i \in N} \left[\pi_i(g) - \overline{\pi}(g)\right]^2 \tag{9}$$

2.3.2. Direct connections and neighborhoods

Computing the average number of neighbors gives us a measure of the network *density*:

$$\overline{\eta}(g) = \frac{1}{\#N} \sum_{i \in N} \eta_i(g) \tag{10}$$

The *range* of the network is a measure introduced by Goyal and Joshi (2002), which is giving the gap between the biggest neighborhood and the smallest one:

$$R(g) = \max_{i \in N} \eta_i(g) - \min_{j \in N} \eta_j(g) \tag{11}$$

Goyal and Joshi (2002) also proposed an index labeled *unequal connections* which measures the average asymmetry between neighborhood size of directly connected people. This index is given by:

$$u(g) = \frac{2}{\sum_{i \in N} \eta_i(g)} \sum_{ij \in g} \left|\eta_i(g) - \eta_j(g)\right| \tag{12}$$

2.3.3. Generic graph properties

Let us introduce the well known indexes of *average path length* and *average cliquishness* introduced by Watts and Strogatz (1998) which are widely used in the physics of networks literature.

The first one is simply computing the average distance of (directly or indirectly) connected agents. It is given by:

$$d(g) \equiv \frac{1}{\#N(\#N-1)} \sum_{i \in N} \sum_{\substack{j \in N \setminus i: \\ \exists i \longleftrightarrow j \subset g}} d(i,j) \tag{13}$$

The *average cliquishness* indicates to what extent the neighborhoods of connected people overlap ("the friends of my friends are my friends"). It is:

$$c(g) = \frac{1}{\#N(\#N-1)} \sum_{i \in N} \sum_{jl: j,l \in N_i(g)} \frac{\Delta(l,j)}{\eta_i(g)} \tag{14}$$

with $\Delta(l,j)$ defines such that $\Delta(l,j) \equiv \begin{cases} 1 \text{ if } j \in N_l(g) \\ 0 \text{ otherwise} \end{cases}$

Finally a very interesting measure of the network structure is the full description of the *degree distribution* (edge distribution over the nodes population): $\{n_k, k = 0, 1, ..., n-1\}$

with $n_k = \sum_{i \in N} \Theta(\eta_i(g), k)$; and $\Theta(\eta_i(g), k) \equiv \begin{cases} 1 \text{ if } \eta_i(g) = k \\ 0 \text{ otherwise} \end{cases}$

2.3.4. Geographic correlation

Lastly, we examine to what extent the geographic distances and the relational connections overlap. Let us thus propose a *geographic correlation* index which gives the geographic distance separating each direct connections in the network:

$$D(g) = \sum_{j:ij \in g} \frac{d'(i,j)}{\eta(g)} \tag{15}$$

3. Dynamic network formation

This section is dedicated to the presentation of our perturbed stochastic process of network formation. We begin with the first step of the dynamic settings, namely the meeting process. We will consider that the probability for a given pair of unconnected agents to be selected is not fixed but varies across agents according to their relative position on the current relational graph. Then, we turn towards the last features of the dynamic process and present its generic properties.

3.1. The preferential meeting process

In most of the works investigating the evolution of network (for example in Watts, 2001; and Jackson and Watts, 2002), it is assumed that any pair of agents have the same probability to meet at each period: it thus constitutes an implicit assumption of an uniform meeting probability: $\forall i, j \in N, p_{ij} = p$. This assumption is twofold: i) every pair of unconnected agents have the same probability to meet; or ii) connected agents reconsider their relations at the same frequency as unconnected ones do.

Here we reject the former part of the assumption while trying to preserve the latter for symmetry reasons. Thus if we write P (Q) the probability that a pair of agents chosen is unconnected (connected), we then assume that:

$$P = \sum_{ij \notin g} p_{ij} = 1 - Q = \frac{\#g^N - \eta(g)}{\#g^N} \tag{16}$$

Together, with considering that $\forall ij \in g, p_{ij} = p$, this implies that the probability that two connected agents meet at each period is such that $p = \frac{1}{\#g^N}$.

Moreover, we do not consider that unconnected people may meet with constant and time independent probabilities. Indeed, this assumption can be justified in the case of anonymous market interactions when the number of agents considered is very large. Here, we introduce a preferential meeting process[10], considering that the probabilities for a pair of unconnected agents to be selected is not independent across agents and vary according to their relative position on the current relational graph. Hence, we consider that the less is the relational distance between two unconnected agents, the greater will be the probability of their selection. Moreover, we consider that this probability increases with their geographic proximity, which is invariant. This ensures that the probability of any two unconnected agents is never null (which is a necessary condition to preserve the ergodicity property of the stochastic process presented below)[11].

Formally, we introduce a preferential meeting process for unconnected agents which is captured by the simple following formula:

$$p_{ij}^{t} = d\left(i,j\right)^{-\gamma} + d'\left(i,j\right)^{-\beta}, \forall ij \notin g_t \tag{17}$$

having introduced time superscripts, and where γ and β are two positive parameters capturing the relative importance of relational indirect connections and geographic proximity in the probability that two unconnected agents meet each other. That expression is also subject to standard normalization, i.e. it is normalized such that: $\sum_{ij \notin g_t} p_{ij}^{t} = \frac{\#g^N - \eta(g_t)}{\#g^N}$.

3.2. The limit behavior of the perturbed stochastic process

The dynamic process can be described as follows. At each time period t, two agents i and $j \in N$ are selected by the preferential meeting process described above. Then, if the selected two agents are directly connected, they can jointly decide to maintain their relation or unilaterally decide to sever the link between them. If they are not connected, they can jointly decide to form a link or renounce unilaterally. Formally, those two situations are the following:

(i) if $ij \in g_t$, the link is maintained if $\pi_i(g_t) \geq \pi_i(g_t - ij)$ and $\pi_j(g_t) \geq \pi_j(g_t - ij)$. Otherwise, the link is deleted.

[10] The notion of "preferential attachment" has been introduced in the model of Albert and Barabási (1999) who rediscovered a process first suggested by Simon (1955). Albert and Barabási show that in real networks, the likelihood of being connecting to a node depends on the number of direct links of this node. However, the preferential attachment process they describe is different of our preferential meeting process since they consider that the networks evolution is built on the addition of new nodes which prefer to be linked to the nodes that have more links. In other words, highly connected nodes increase their connectivity faster than their less connected peers. For quantitative support on the presence of preferential attachment, one can refer to Jeong *et al.* (2003) who provide some measurements on four networks (science citation network, WWW, actor collaboration and science co-authorship network).

[11] In a similar way (and for a similar reason), Vega-Redondo (2002) considers two possible "routes of search" in the links formation process. The first is local that is mediated by the social network while the second is said to be "global" since the meeting occurs between agents in two different components of the network with a small probability.

(ii) if $ij \notin g_t$, a new link is created if $\pi_i(g_t + ij) \geq \pi_i(g_t)$ and $\pi_j(g_t + ij) \geq \pi_j(g_t)$,with a strict inequality for one of them.

The stochastic process introduced here can be defined as a Markov chain which finite states correspond to the "current" network at the end of a given period. In other words, the state of the system at time t (with $t = 0, 1, 2, ...$) is given by the graph structure $g_t \in G$. The evolution of the system $\{g_t, t \geq 0\}$ can be described as a discrete-time stochastic process with state space G.

Following Jackson and Watts (2002), we then introduce small random perturbations ε ($\varepsilon \in (0, a]$) which invert agents' right decisions in creating, maintaining or deleting links. These perturbations may be understood as mistakes or as mutations. The characterization of the asymptotic behavior of this process is due to Young (1993). For small but non null values of ε ($\varepsilon \in (0, a]$), it can be shown that the discrete-time Markov chain being irreductible and aperiodic, has a unique corresponding stationary distribution. Such perturbed stochastic processes are said to be ergodic. Intuitively ergodicity implies that it is possible to transit directly or indirectly between any chosen pair of states in a potentially very long period of time (which also means that any state of the system can be directly or indirectly reached from any given one)[12]. Moreover, when ε goes to zero, the stationary distribution converges to a unique *limiting stationary distribution*. The states that are in the support of this limiting stationary distribution are called *stochastically stable and* are either pairwise stable (cf. Definition 1) either part of a close cycle of states[13]. Notice that the ergodicity property is quite interesting since it allows us to run numerical simulations in order to examine the long run behavior of the system (Vega-Redondo, 2002): we can then compute the unique limiting stationary distribution of the process.

4. Networks selection in the simple collective innovation model: the results

In this section, we present the results obtained for a simplified version of the collective innovation model presented in Section 2. As it is explained in Remark 1, our general model (7) may be simplified to obtain the following specification of the profit function which from now on becomes our basic profit equation:

$$\pi_i(g_t) = \sum_{j \in N \setminus i} \delta^{d(i,j)} - c \sum_{j: ij \in g_t} d'(i,j) \tag{18}$$

[12]It allows the long run state of the system to become independent of its initial conditions. Indeed, processes that are non-ergodic are said to be "path dependent" (David, 1985) since their limiting behavior is dependent on the initial state of the system.

[13]Such process is called a regular perturbation of the initial stochastic process (without trembles). Definitions, properties and some proofs are examined in Carayol and Roux (2003b). Initial contributions are the ones of Freidlin and Wentzell (1984), Young (1993), Kandori *et al.* (1993), and Jackson and Watts (2002).

Let recall that in this model, agents ares located on a circle. Moreover, for simplification purposes, we will consider that $c = \frac{2}{n}$ for even values of n, and $c = \frac{2}{n-1}$, otherwise[14]. Recall also that the dynamic process used is based on the preferential meeting principle introduced in Section 3. For simplification purposes again, we use a simple rule assuming that $\gamma = \beta = 1$. Meeting rule before normalization (17) then becomes:

$$p_{ij}^t = \frac{1}{d(i,j)} + \frac{1}{d'(i,j)}, \forall ij \notin g_t \tag{19}$$

We next propose to numerically simulate the unique limiting stationary distribution of the perturbed dynamic process of Jackson and Watts (2002) (for which the error term is decreasing down to zero) by the following simple rule:

$$\varepsilon^t = \begin{cases} 0.02 & \text{if } t < 50 \\ 1/t & \text{otherwise} \end{cases} \tag{20}$$

Thus we ensure that errors affect the dynamics while it is decreasing down to zero when time increases: $\lim_{t\to\infty} \varepsilon^t = 0$.

In the following subsection we study the limit distribution of states and show that *small world*-like networks may be selected through the process of network formation. Secondly, we propose a more systematic analysis of how network architecture varies with the decay parameter δ, that is how network selection is affected by the easiness of knowledge flows on networks.

4.1. Limit networks selection: the emergence of small worlds

The first goal is to study the limit distribution of the process in one simple numerical situation. For that purpose, we ran $1,000$ simulations of $10,000$ periods[15] with the empty graph as initial condition and with $\gamma = \beta = 1, \delta = 0.7$ and $c = 0.1$. Nodes degree distribution is presented in Figure 1. One can observe that the distribution peaks at 6 neighbors, being slightly asymmetric. It is to be noticed that no agent has less that four neighbors: this is because establishing direct links with geographically close agents is weakly costly. In the meantime, there is no agent having more than eight neighbors because no-one is intending to support the high costs of many direct links. The network self-organizes itself in a shape which has some in common with regular networks. One may observe in the descriptive statistics obtained on such distribution (presented in Table 1 in the Appendix) that the cliquishness coefficient $c(g)$ is quite high: nearly as high as the one of the double ring $g^{2\circ}$. More, these clustered networks are correlated to the geographic metric: the average geographic distance between

[14] This simplification is close to the one introduced by Johnson and Gilles (2000) in the "line world" case.

[15] Time series analyses conduced over more than 100.000 periods showed that the process has nearly allways converged on a given pairwise stable state after 10.000 periods. For evidence and details see a companion paper Carayol and Roux (2003b).

connected pairs of agents is quite small ($D(g) \simeq 2.5$). However, the network departs from such regular structure in that the average path length is singularly lower than for the single ring ($1.84 < d(g^{\circ}) \simeq 5.26$).

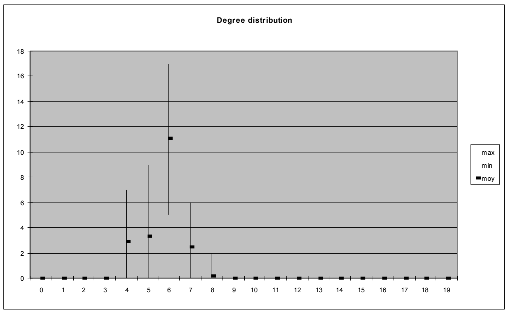

Figure 1. *The nodes degree distribution (mean-max-min) in the simple collective innovation model (with 1000 experiments,* 20 *agents,* 10,000 *periods).*

In order to provide a better understanding of these results, we represent in Figure 2 below two networks structures selected in the long run. The first structure is obtained with our model whereas the second is obtained with the Simple Connections model of Jackson and Wolinski (1996), that is both without preferential meeting and without link costs increasing with geographic distance. It should be noticed that both networks are pairwise stable for their respective payoffs function. The left graph clearly exhibits small world features: high clusterization while some distant connections remain. How such stable small world has been selected from the empty network is presented in the appendix (Figures 6, 7 and Table 2).

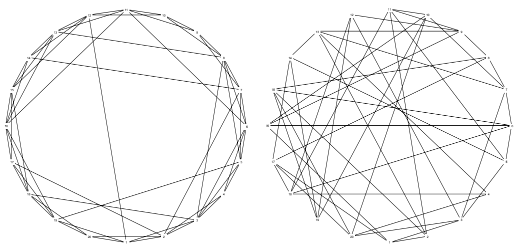

Figure 2. *Two typical selected networks after 10,000 periods respectively with (left graph) and without (right graph) preferential meeting and geo link costs.*

4.2. How the limit distribution varies with delta?

In the following, we study how the limit distribution varies with the decay parameter δ. To do so we performed a new set of experiments of 10,000 periods, but now we perform 50 experiments for each small increment of δ over its value space ($]0, 1[$). These results are exposed in Figure 1. We find that the average efficiency increases with δ. Average neighborhoods size exhibits an inverse U-shape. The average path length decreases from $\delta = 0.1$ until $\delta = 0.4$, then remains nearly constant and increases for high values of δ (≥ 0.75).Average cliquishness suddenly increases from zero at $\delta = 0.2$ and reaches his maximum again very rapidly for $\delta = 0.35$. Then it decreases slowly and stabilises until $\delta = 0.75$, from where it goes down to 0 when δ is close to 1. Then it decreases slowly and stabilises until $\delta = 0.75$, from where it goes down to 0 when δ is close to 1. Geographic correlation index increases until $\delta = 0.5$ and decreases from $\delta = 0.8$. Finally, we surprisingly observe that the more δ the more the instability of the graph: activity (creation plus deletion) increases with δ up to 0.85. The intuition for this result is that the less δ, the less substitute are direct (close vs. distant) links. These results are exposed in the Figure 3.

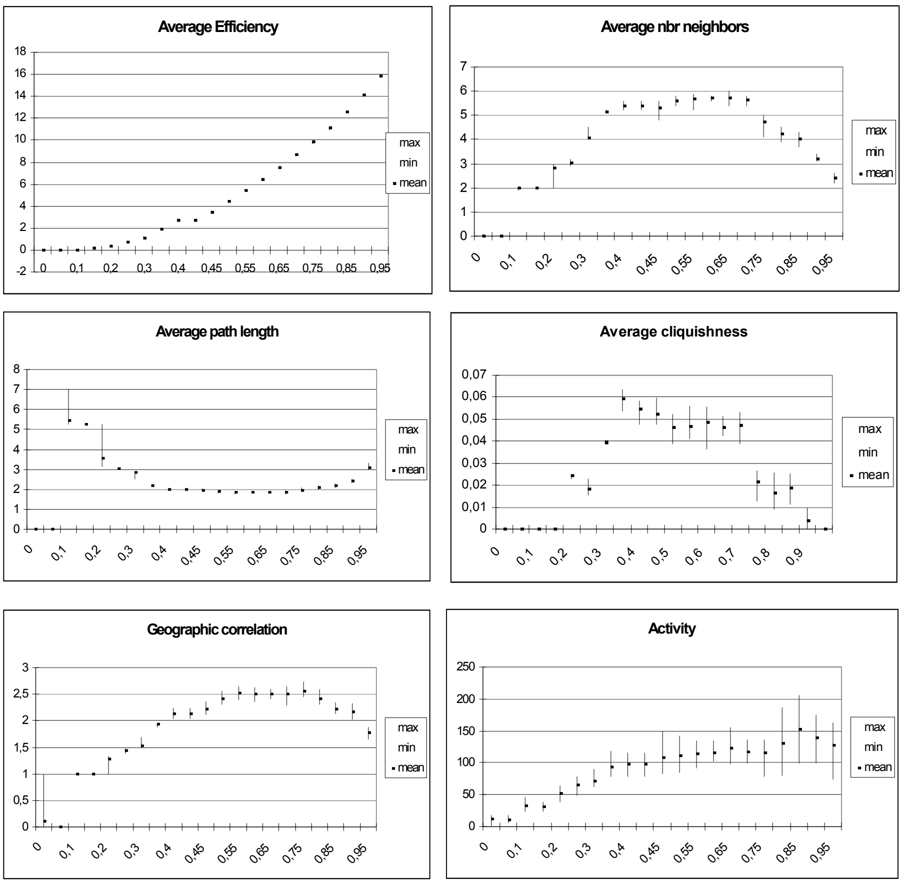

Figure 3. *The graph indexes in the simple collective innovation model when $\delta \in]0, 1[$ varies (1000 experiments, 20 agents, 10,000 periods).*

The different networks configurations that emerge in the long run are more easily observed by degree distribution presented in the Figure 4. Therein, we can (indirectly) observe that the empty graph $g^{\emptyset}$ is selected when $\delta \leq c = 0.1$. When $c < \delta \leq 2c$ the geographic ring g° emerges: in this case, all agents are connected to their two closest neighbors. When δ is 0.3, nearly all agents are connected to four agents who are likely to be their four closest geographic neighbors. This situation corresponds to the double geographic ring $g^{2\circ}$. From $0.4 \leq \delta \leq 0.7$, we observe a very 'stable' situation (plateau) characterized by flat maximum neighborhood sizes which decrease from there. At the very beginning of that configuration ($\delta \simeq 0.35$), we already have the weakest average path length while average cliquishness is still close to its maximum (cf. Figure 3). Such a situation presents many similarities with the small world network structure. While the 'plateau' configuration is exposed in the first graph of Figure 2, the other typical ones may be found in Figure 5.

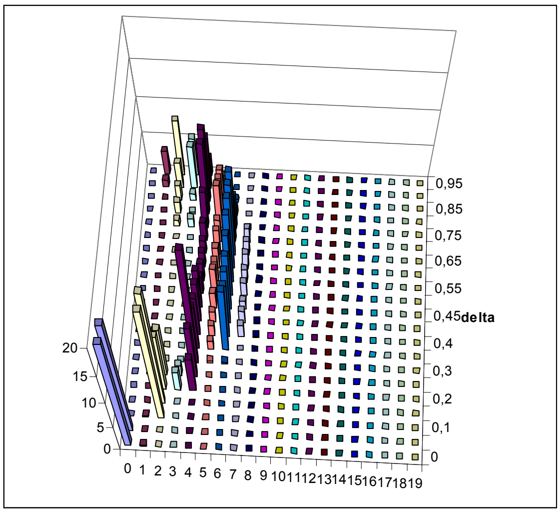

Figure 4. *The limit degree distribution in the simple collective innovation model when* $\delta \in]0, 1[$ *varies (1000 experiments,* 20 *agents,* 10,000 *periods).*

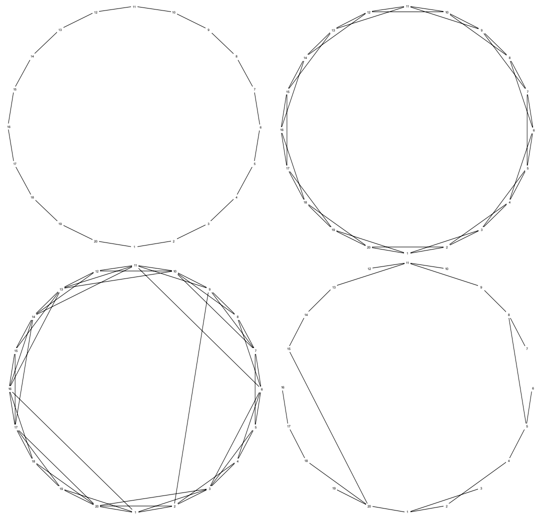

Figure 5. *Limit typical stable networks selected by the stochastic process in the simple collective innovation model (with* 20 *agents,* 10,000 *periods). The first network (simple ring* g°*) has been obtained for* $\delta = 0.1$; *the second network (double ring* $g^{2\circ}$*) has been obtained for* $\delta = 0.3$; *the third network (small world) has been obtained for* $\delta = 0.35$; *the last network has been obtained for a high value of delta* $\delta = 0.98$.

5. Conclusion

In this paper, we examined a dynamic stochastic process of network formation following the contribution of Jackson and Watts (2001) who introduced a dynamic stochastic process in the initial model of network formation proposed by Jackson and Wolinski (1996). In our network based model of 'collective innovation', agents benefit from knowledge flows by communicating with agents with whom they are directly or indirectly connected. To examine the selected innovation networks, we studied the characteristics of the long term selected graphs, without limiting our attention to some typical structures (the empty graph, the star or the complete network). We made use of several statistical indexes to capture interesting features of the graphs (average path length, clustering coefficient, node degree distribution, etc.). We studied the stability against noise (error probability) of the graphs selected and also computed their efficiency. We also introduced heterogenous cost of linking and a preferential meeting rule governing the dynamic process of links formation, which consisted in weighting the meeting probability between any two agents by the inverse of their relational and geographic distance. The results concerned the set of stochastically stable networks selected. For different numerical values of the knowledge transferability parameter (decay parameter δ), we described the different network architectures that are emerging in the long run.

These results may contribute to shed a new light on the issue of networks formation which proves to be crucial for the distributed innovation phenomenon. We showed that when the transferability of knowledge is low, networks tend to be locally clustered. Increasing slightly the transferability of knowledge increases the density of local networks. Increasing again slightly that value stimulates the emergence of distant connections. This last result consists in computing the critical values of the decay in knowledge spillover for which pairwise stable small worlds networks are dynamically selected.

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Appendix

	mean	median	max	min	var
Av efficiency: $\overline{\pi}$	8,70	8,70	8,87	8,57	0,00
Var efficiency: $var(\pi)$	27353	27344	85943	28412	26560
Av nbr neighbors: $\overline{\eta}(g)$	5,67	5,70	6,20	5,3	0,03
Min nbr neighbors: $\min \eta_j(g)$	4,04	4,00	5	4	0,04
Max nbr neighbors: $\max \eta_i(g)$	7,06	7	8	6	0,17
Range neighbors: $R(g)$	3,02	3	4	1	0,22
Unequal connections: $u(g)$	1,84	1,82	3,21	0,55	0,21
Av path length: $d\,(g)$	1,84	1,84	1,94	1,69	0,00
Av cliquishness: $c\,(g)$	0,042	0,042	0,053	0,03	0,00
Geo correlation: $D(g)$	2,47	2,46	2,77	2,26	0,01
Activity	116,15	115	171,78	78	218,1

Table 1. *Some descriptive statistics on the graph indexes computed for the limit graph distribution in the collective innovation model.*

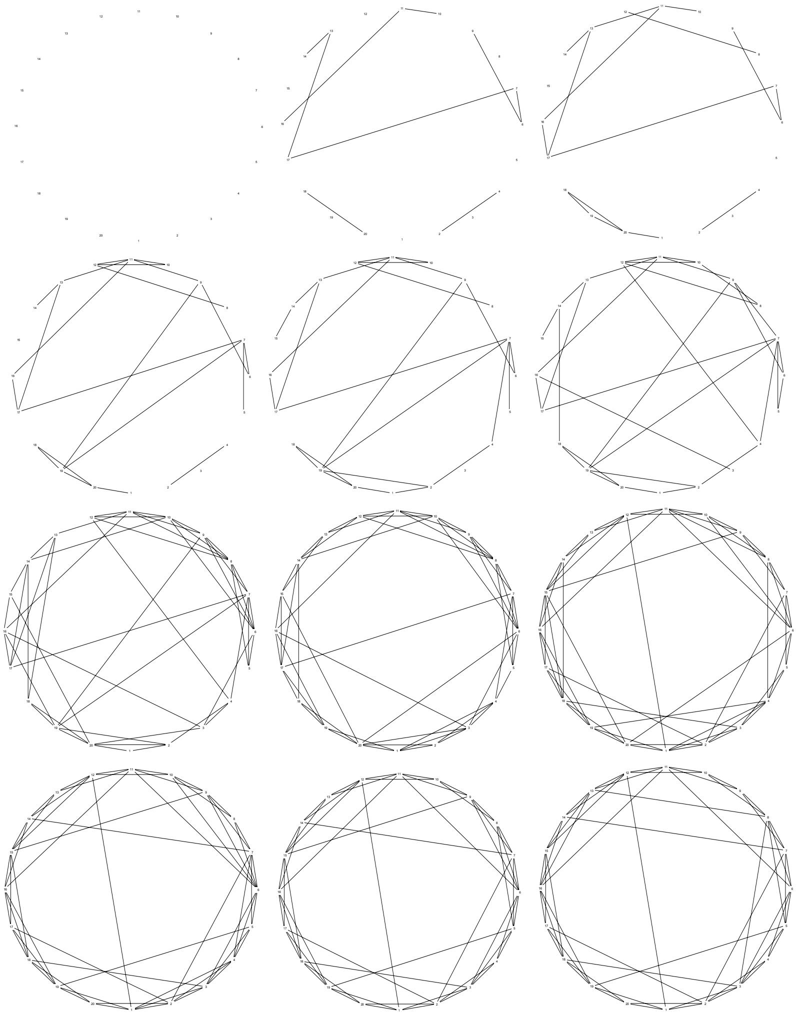

Figure 6. *An example of stable network formation in the collective innovation model.*

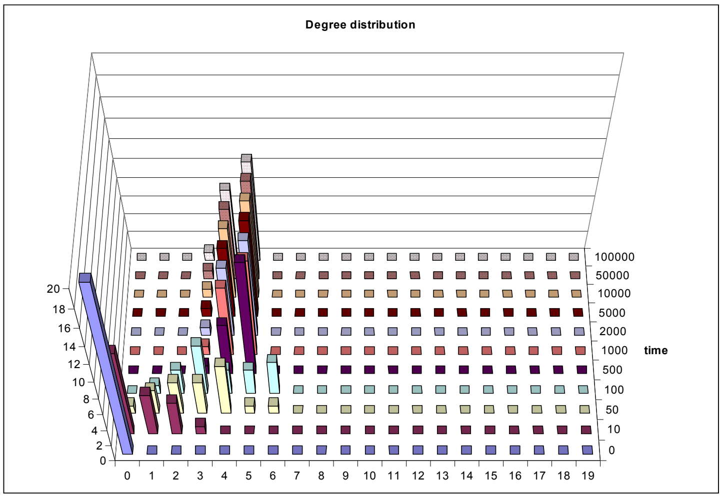

Figure 7. *The degree distribution of the network presented in Figure 6.*

Time	Efficiency	Nbr links	Creations	Deletions	Errors	Pairwise stability
0	0	0	0	0	0	0
10	15,05	9	9	0	0	0
20	34,09	15	15	0	0	0
30	84,59	21	21	0	1	0
40	115,35	25	25	0	2	0
50	136,94	31	31	0	2	0
100	162,81	48	49	1	5	0
200	169,43	54	62	8	6	0
500	171,96	58	79	21	6	0
1000	173,13	57	84	27	6	0
2000	172,81	55	86	31	6	1
5000	172,81	55	86	31	6	1
10.000	173,31	56	91	35	7	1

Table 2. *Some graph indexes of the network presented in Figure 6.*